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Effective Conductivity of a Hexagonal Network

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The hexagonal network is an important optimal composite structure. The thermal diffusion (mass diffusion, electric conduction) through a hexagonal network of finite thickness is analyzed for the first time. We assume the network has finite conductivity, but the inclusions enclosed by the network are either insulating or perfectly conducting. Using eigenfunction expansions and matching three different subregions, the temperature profiles and the effective conductivity are found. The special case of circular inclusions agrees with the existing literature. Junction effects, which introduce additional resistances to the simple electric circuit analogy, are determined.

Nomenclature

A_n, B_n, C_n	=	constants
a	=	fractional length of straight section
f, g, h	=	functions describing curved section
k	=	conductivity
k^*	=	effective conductivity
L	=	half width of straight section
N	=	number of terms
n	=	integer
Q	=	heat flux
r	=	cylindrical coordinate
T	=	normalized temperature
x, y	=	Cartesian coordinates
α_n	=	$(n-0.5)\pi$
γ_n	=	$n\pi/2$
ε	=	equivalent length
θ	=	cylindrical coordinate
σ	=	k^*/k

Subscripts

j	=	integer index
I, II, III	=	regions of the network
IV, V	=	regions of the voids or inclusions

Introduction

WE find in nature an abundance of hexagonal structures, including the honeycombs of bees and ants, colonies of sponges and corals, and bone. Man, in an effort to optimize, produced hexagonal structures such as those used in convective cooling, artificial honeycombs, and composite materials for light weight and strength.

A hexagonal network (Fig. 1a) contains hexagonal voids or inclusions whose property differ from the network itself. Thus the system is a composite of two material phases. It would be essential to determine the effective property of a composite from its constituents. The most accurate method is to solve the governing equations for each material micromechanically, then derive its collective property. One of the most important property is effective conductivity [1]. Methods to study conductivity include the expansion of singularities [2,3], boundary collocation [4], finite differences [5,6], or eigenfunction expansions and matching [7,8]. Most literature considered circular inclusions although square [4,9] and rectangular [7,8] inclusions have also been investigated. For hexagonal inclusions studied in this paper, the applicable methods are finite differences (or finite elements) and eigenfunction expansions and matching. We choose the latter, which is more efficient in comparison.

In what follows, we shall study the diffusion through a hexagonal network. Although the terminology used is in terms of thermal diffusion, the results also apply to mass diffusion and electric conduction as well.

We assume the conductivity of the spaces enclosed by the network differs greatly from that of the network itself, that is, one can consider the "voids" are either insulating or perfectly conducting.

Formulation

Let us consider the case where mean heat flux Q (per area) is parallel to one branch of the network (Fig. 1a). Let L be the half width of the branches. The center lines of each branch then enclose hexagonal tiles. Each tile has an edge length $2(\sqrt{3}+a)L$ where a is a positive constant. Let T_1 and T_0 be the mid branch temperatures as shown. Let T' be the dimensional temperature related to the normalized temperature T by $T=(T'-T_0)/(T_1-T_0)$. Because there is symmetry, we can study the small section enclosed by the dashed lines in Fig. 1a. Figure 1b, now normalized by L, shows this section enlarged and further divided into three regions, each with their own coordinates. Regions I and III are rectangular, whereas region II is irregular, but close to and bounded by the semicircle of radius 2. Region IV is the hexagonal void adjacent to the upper boundary of the network in Fig. 1b and region V is that adjacent to the left boundary.

First consider the case where the hexagonal voids are highly conducting. It is clear that, due to symmetry, the temperature on a plane through the center of the hexagon and normal to the flux Q must be constant. That constant is $T=1$ for region IV in Fig. 1b. On the other hand, the origin of region I (where $T=0$) is an antisymmetry point. Thus region V must have temperature $T=-1$. Actually this property is a special case of antipolar symmetry, that is, $T(x,y)=-T(-x,-y)$ about the origin $T(0,0)=0$. We conclude along the upper phase boundary $T=1$ and on the left phase boundary $T=-1$. The general solution for region I satisfying the Laplace equation, $T(x,\pm 1)=\pm 1$ and the antipolar symmetry condition is

$$T_{\mathrm{I}}(x,y)=y+\sum_{1}^{2N}A_n[e^{\gamma_n(x-a)}+(-1)^n e^{-\gamma_n(x+a)}]\sin[\gamma_n(1+y)] \tag{1}$$

where A_n are coefficients to be determined, $\gamma_n=n\pi/2$, and we have truncated the series to $2N$ terms. We comment that the deduction of this expression is not trivial. The general solution for region II, with symmetry about $\theta=0$, π is

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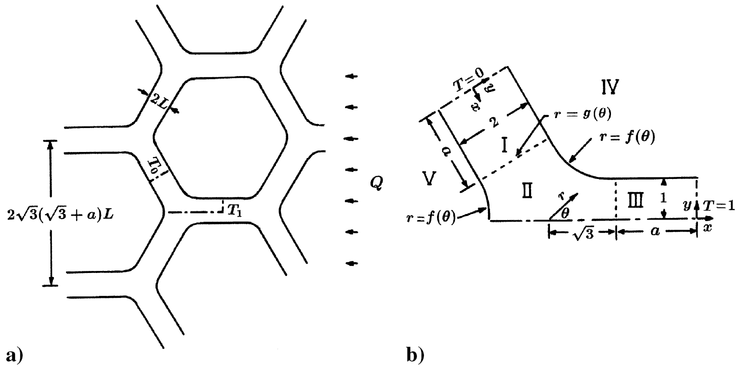

Fig. 1 a) The hexagonal network. b) Regions I–III in a repeating section and regions IV, V of the inclusion.

$$T_{\mathrm{II}}(r,\theta) = C_0 + \sum_{1}^{6N-1} C_n \left(\frac{r}{2}\right)^n \cos(n\theta) \tag{2}$$

There are $6N$ unknown coefficients. The general solution for region III, having antisymmetry in x and symmetry in y is

$$T_{\mathrm{III}}(x,y) - 1 = B_0 x + \sum_{1}^{N-1} B_n[e^{\alpha_n(x-a)} - e^{-\alpha_n(x+a)}]\cos(\alpha_n y) \tag{3}$$

where $\alpha_n = (n-0.5)\pi$.

The next step is to point-match on the boundary of region II to satisfy the boundary conditions and match between regions. Choose $6N$ equally spaced angles between $[0,\pi]$:

$$\theta_j = \frac{(j-0.5)}{6N}\pi, \qquad j = 1 \text{ to } 6N \tag{4}$$

For the common boundary between region II ($r = \sqrt{3}/\cos\theta$) and region III ($x = -a$); we require the temperature be continuous

$$T_{\mathrm{III}}(-a, \sqrt{3}\tan\theta_j) = T_{\mathrm{II}}\left(\frac{\sqrt{3}}{\cos\theta_j}, \theta_j\right) \qquad j = 1 \text{ to } N \tag{5}$$

and also their normal derivatives be continuous. In general the normal derivative to a surface $r = h(\theta)$ is

$$\frac{\nabla(r-h)}{|\nabla(r-h)|}\cdot\nabla T = [1 + \langle h'(r)\rangle^2]^{-\frac{1}{2}}\left[\frac{\partial T}{\partial r} - \frac{h'}{r^2}\frac{\partial T}{\partial\theta}\right] \tag{6}$$

Because $h = \sqrt{3}/\cos\theta$ on this boundary the continuity of normal derivatives becomes

$$\frac{\partial T_{\mathrm{III}}}{\partial x}(-a, \sqrt{3}\tan\theta_j) = \left.\frac{\partial T_{\mathrm{II}}}{\partial x}\right|_{x=\sqrt{3}} = \left[\cos\theta_j \frac{\partial T_{\mathrm{II}}}{\partial r} - \frac{\sin\theta_j}{r}\frac{\partial T_{\mathrm{II}}}{\partial\theta}\right]_{r=\sqrt{3}/\cos\theta_j}$$
$$j = 1 \text{ to } N \tag{7}$$

On the corner section $r = f(\theta)$ on upper right $T_{\mathrm{II}} = 1$ or

$$T_{\mathrm{II}}[f(\theta_j), \theta_j] = 1 \qquad j = N+1 \text{ to } 3N \tag{8}$$

Here $f(\theta)$ can be any reasonable function describing the corner region. We considered two cases: a straight line corner where

$$f_1(\theta) = \frac{2}{\sin\theta + \cos\theta/\sqrt{3}} \tag{9}$$

and a rounded corner with radius of curvature 2,

$$f_2(\theta) = \sqrt{3}\cos\theta + 3\sin\theta - \sqrt{(\sqrt{3}\cos\theta + 3\sin\theta)^2 - 8} \tag{10}$$

The boundary between regions I and II is at

$$r = g(\theta) = \frac{2}{\sin\theta - \cos\theta/\sqrt{3}} \tag{11}$$

We require continuity of temperatures and their normal derivatives. Using Eq. (6);

$$T_{\mathrm{I}}[a, \sqrt{3}\tan(2\pi/3 - \theta_j)] = T_{\mathrm{II}}[g(\theta_j), \theta_j] \qquad j = 3N+1 \text{ to } 5N \tag{12}$$

$$-\frac{\partial T_{\mathrm{I}}}{\partial x}[a, \sqrt{3}\tan(2\pi/3 - \theta_j)] = \frac{1}{2}\left[(\sqrt{3}\sin\theta_j - \cos\theta_j)\frac{\partial T_{II}}{\partial r} + \frac{1}{r}(\sqrt{3}\cos\theta_j + \sin\theta_j)\frac{\partial T_{\mathrm{II}}}{\partial\theta}\right]_{r=g(\theta_j)}$$
$$j = 3N+1 \text{ to } 5N \tag{13}$$

Last, on the left side boundary $r = f(\theta)$, $T_{\mathrm{II}} = -1$ or

$$T_{\mathrm{II}}[f(\theta_j), \theta_j] = -1 \qquad j = 5N+1 \text{ to } 6N \tag{14}$$

Equations (5), (7), (8), and (12–14) represent a system of $9N$ linear equations that can be inverted by standard means for the $9N$ unknowns: $A_1, \ldots A_{2N}$, $B_0, \ldots B_{N-1}$, $C_0 \ldots C_{6N-1}$. In general, the error is less than 1% if N is larger than 10. Figure 2 shows typical isotherms computed for the repeating section.

Next, consider the insulated case or when the voids are almost nonconducting. The general solution for region I satisfying antipolar symmetry and zero normal derivatives on $y = \pm 1$ is

$$T_{\mathrm{I}}(x,y) = A_0 x + \sum_{1}^{2N-1} A_n[e^{\gamma_n(x-a)} - (-1)^n e^{-\gamma_n(x+a)}]\cos[\gamma_n(1+y)] \tag{15}$$

The solution for region II is the same as Eq. (2). The solution for region III is the same as Eq. (3), except now $\alpha_n = n\pi$. The boundary conditions are similar, except Eq. (8) is replaced by

$$\left[\frac{\partial T_{\mathrm{II}}}{\partial r} - \frac{f'(\theta_j)}{r^2}\frac{\partial T_{\mathrm{II}}}{\partial\theta}\right]_{r=f(\theta_j)} = 0 \qquad j = N+1 \text{ to } 3N \tag{16}$$

and Eq. (14) is also replaced by Eq. (16), with $J = 6N+1$ to $6N$. The $9N$ linear equations are solved for the two types of corner shapes Eq. (9) or Eq. (10). Typical isotherms are shown in Fig. 3.

Effective Conductivity

First consider the perfectly conducting inclusions case. The total heat in one period is equal to heat across the center line of the two angled branches.

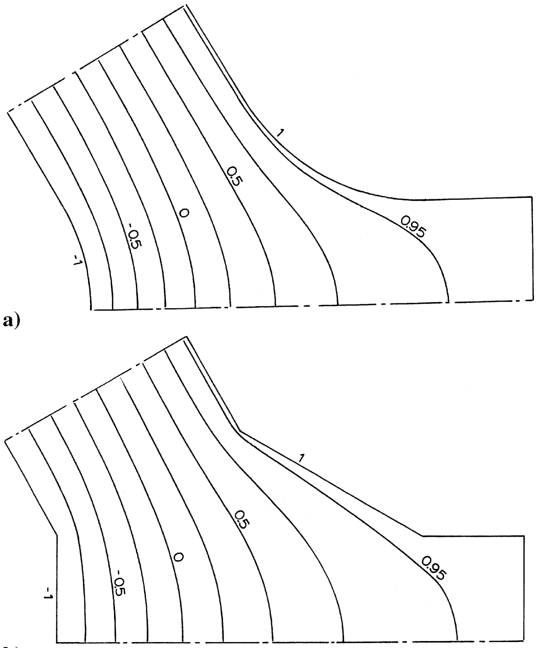

Fig. 2 Isotherms for the perfectly conducting case. a) rounded corner b) straight corner.

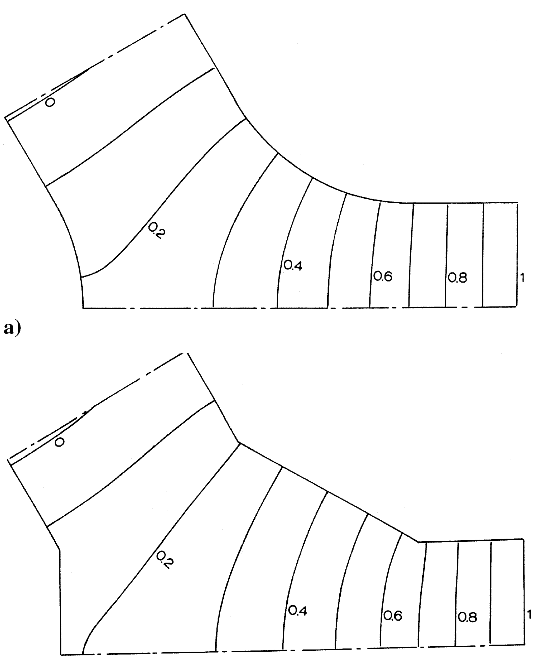

Fig. 3 Isotherms for the perfectly insulated case. a) rounded corner b) straight corner.

$$Q 2\sqrt{3}(\sqrt{3}+a)L = 4k(T_1 - T_0)\left[\int_0^a \frac{\partial T_{\mathrm{I}}}{\partial y}\bigg|_{y=0} \mathrm{d}x - \int_0^{\sqrt{3}} \frac{1}{r}\frac{\partial T_{\mathrm{II}}}{\partial \theta}\bigg|_{\theta=2\pi/3} \mathrm{d}\theta\right] \tag{17}$$

But Q is also equal to the effective conductivity times the temperature gradient between the origins of regions I and III,

$$Q = k^* \frac{\Delta T}{\Delta x} = k^* \frac{(T_1 - T_0)}{3(\sqrt{3}+a)L/2} \tag{18}$$

Let σ be the ratio of effective conductivity to conductivity of the network. Using Eqs. (17) and (18)

$$\sigma = \frac{k^*}{k} = \sqrt{3}\left[\int_0^a \frac{\partial T_{\mathrm{I}}}{\partial y}\bigg|_{y=0} \mathrm{d}x - \int_0^{\sqrt{3}} \frac{1}{r}\frac{\partial T_{\mathrm{II}}}{\partial \theta}\bigg|_{\theta=2\pi/3} \mathrm{d}r\right] = \sqrt{3}\Bigg[a + \sum_1^{2N} A_n \cos(\gamma_n)\langle 1 - (-1)^n e^{-2\gamma_n a}\rangle + \sum_1^{6N-1} C_n \left(\frac{\sqrt{3}}{2}\right)^n \sin\left(\frac{3\pi n}{3}\right)\Bigg] \tag{19}$$

For $a = 0$, regions I and III are absent. In the case of a rounded corner [Eq. (10)] the inclusion becomes a circle. The geometry becomes a composite with an triangular array of circular cylinders. The volume fraction is $2\pi/9\sqrt{3} = 0.4031$. For the perfectly conducting case, our computation [Eq. (19)] converges to $\sigma = 2.3512$. The results of Perrins et al. [2] and Lu [4], after interpolation to the same volume fraction, gives a value of 2.3546, which is within 0.2% of our result.

For the insulated inclusions case, the total heat in one period is equated to that passes through region III, or

$$Q 2\sqrt{3}(\sqrt{3}+a)L = 2k(T_1 - T_0)\int_0^1 \frac{\partial T_{\mathrm{III}}}{\partial x}\bigg|_{x=0} \mathrm{d}y \tag{20}$$

Together with Eq. (18) we find

$$\sigma = \frac{k^*}{k} = \frac{\sqrt{3}}{2}\int_0^1 \frac{\partial T_{\mathrm{III}}}{\partial x}\bigg|_0 \mathrm{d}y = B_0 \tag{21}$$

Again, for $a = 0$ the inclusions become circles. We obtained a value of $\sigma = 0.4252$, which is less than 0.1% of the value of 0.4255 interpolated from Lu [4].

The results for other values of a are listed in Table 1. According to Keller's reciprocal theorem [10], the product of $\sigma_1\sigma_2$ and $\sigma_3\sigma_4$ are to be unity. Our products are within 0.2% of unity, thus further ascertains the accuracy of our method.

Approximate Formulas for Large a

When a is large, the inclusion area becomes more hexagonal and the network becomes long strips connected at ends. For the perfectly conducting case most of the heat transfer occurs between the two sides of region I. Assuming a linear temperature distribution, we find $\sigma = \sqrt{3}a$. Let the contribution of region II be equivalent to an added ε length to a, that is,

$$\sigma' = \sqrt{3}(a + \varepsilon) \tag{22}$$

Using the large a numerical values of Table 1, we find for the rounded corner $\varepsilon_1 = 1.3510$ and thus the approximate formula

$$\sigma_1' = \sqrt{3}(a + 1.351) \tag{23}$$

Similarly, for the straight corner $\varepsilon_3 = 1.1735$, thus

$$\sigma_3' = \sqrt{3}(a + 1.174) \tag{24}$$

For the insulated case most of the heat transfer is along the branches. Ignoring region II and assuming a linear longitudinal gradient, the first approximation is $\sigma = 1/(\sqrt{3}a)$. Let region II be represented by an added length ε. Then

$$\sigma' = \frac{1}{\sqrt{3}(a + \varepsilon)} \tag{25}$$

From the data of Table 1, we deduce the approximate formulas

$$\sigma_2' = \frac{1}{\sqrt{3}(a + 1.352)} \tag{26}$$

Table 1 Effective conductivity ratios. σ_1 rounded corner, perfectly conducting inclusion; σ_2 rounded corner, insulated inclusion; σ_3 straight corner, perfectly conducting inclusion; σ_4 straight corner, insulated inclusion. The prime denotes values from approximate formulas

a	σ_1	σ_1'	σ_2	σ_2'	σ_3	σ_3'	σ_4	σ_4'
0	2.351	2.340	0.4252	0.4270	2.041	2.033	0.4869	0.4893
0.25	2.779	2.773	0.3598	0.3604	2.4771	2.4664	0.4027	0.4037
0.50	3.209	3.206	0.3116	0.3117	2.907	2.899	0.3433	0.3437
1	4.073	4.072	0.2455	0.2455	3.766	3.765	0.2649	0.2648
2	5.805	5.804	0.1723	0.1722	5.497	5.498	0.1816	0.1816
5	11.001	11.00	0.0909	0.0909	10.693	10.69	0.0934	0.0934
10	19.661	19.66	0.0509	0.0509	19.353	19.35	0.0516	0.0516

$$\sigma_4' = \frac{1}{\sqrt{3}(a + 1.180)} \tag{27}$$

We see from Table 1 the symptotic formulas are good approximations for $a \geq 0.5$. Although $\sigma_3'\sigma_4'$ is not exactly unity, it is within an error of 0.5%.

The approximate formulas also show the effect of the junction becomes independent of branch length a for $a \geq 0.5$. Thus one can introduce the concept of junction conductance (for the perfectly conducting case) and junction resistance (for the insulated case) in terms of added lengths ε. This ε is a function of the junction shape only. Thus one can easily predict the conductance or resistance of a general network using bifurcating junctions and different branch lengths a.

Discussions

Our method of eigenfunction expansions and matching worked well. If numerical finite differences were used, one needs to guess and iterate for the unknown inlet and exit temperatures, which is quite tedious.

We also obtained detailed temperature profiles. Notice that for the same geometry the isotherms for the insulated case and those of the perfectly conducting case are not orthogonal. This is because the boundary conditions at the inlet (and exit) surfaces are not orthogonal.

We have considered the effective conductivity in a direction parallel to one of the branches. In general, the effective conductivity is a tensor. However, due to the regular hexagonal symmetry, it can be shown (Wang [1]) that the effective conductivity is transversely isotropic. This means the value of the effective conductivity is the same for any transverse mean flux direction. We mention that the longitudinal effective conductivity can be obtained easily by the law of mixtures.

Our method can be applied to other junction shapes $f(\theta)$. In general, convergence is faster when region II is closer to a circle. For the shapes considered in the present paper, $N = 12$ is adequate for a 3-figure accuracy.

In the case the conductivities of the network and the inclusions are of the same order, such that the assumption of insulating or perfectly conducting inclusions does not apply, regions IV, V representing the inclusion are to be matched with the network. The inclusions are almost circular and polar coordinates are appropriate. Such cases are not considered here but the method is similar.

In network modeling, branches are usually replaced by resistance elements and the effect of the junction itself is ignored. We find the effect of the junction is equivalent to an added length ε, which is of order one. Thus the branch length a needs to be larger than 100 (length to width of branch) to ignore the junction with an error $\leq 1\%$. For shorter branch lengths one should include junction resistances (or conductances) found in this paper.

The importance of hexagonal networks in composite structures cannot be overestimated. The present paper presents the first theoretical determination of the conductivity of a hexagonal network. The effective conductivities listed in Table 1 and their approximate formulas should be quite useful in predicting heat or mass diffusion through hexagonal networks. We hope that this work will elicit further experimental and theoretical studies on this important topic.

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